

POISSON MANIFOLDS AND THEIR ASSOCIATED STACKS

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ABSTRACT. We associate to any integrable Poisson manifold a stack, i.e. a category fibered in groupoids over a site. The site in question has objects Dirac manifolds and morphisms pairs consisting of a smooth map and a closed 2-form. We show that two Poisson manifolds are symplectically Morita equivalent if and only if their associated stacks are isomorphic.

1. INTRODUCTION

In differential geometry, differentiable stacks provide models for singular spaces. Intuitively, a differentiable stack generalizes the notion of manifold, where atlas are replaced by Lie groupoids. The stack itself can be thought of as the orbit space of a Lie groupoid.

In the study of morphisms and isomorphisms of differentiable stacks one uses three, roughly equivalent, languages. Firstly, one can define a *generalized morphism* of Lie groupoids to be a left principal bibundle. This is built upon work by Haefliger [12], Moerdijk [16], Mrčun [17], and others. Another approach is to localize the category of Lie groupoids at special homomorphisms called *weak equivalences* (see [11] for a good exposition). Lastly, we can take the approach of Grothendieck and consider fibered categories over a site. In this setting we replace a Lie groupoid with its associated category of principal bundles. We summarize these points of view in the following table:

Principal Bibundles	<i>objects</i> : Lie groupoids <i>morphisms</i> : left principal bibundles <i>equivalences</i> : principal bibundles
Calculus of Fractions	<i>objects</i> : Lie groupoids <i>morphisms</i> : formal factions $\frac{F:\mathcal{G}'\rightarrow\mathcal{G}}{G:\mathcal{G}'\rightarrow\mathcal{H}}$ <i>equivalences</i> : a pair of weak equivalences F/G
Fibered Categories	<i>objects</i> : categories fibered in groupoids <i>morphisms</i> : fiber preserving functors <i>equivalences</i> : equivalence of categories

In [20], Ping Xu introduced a notion of Morita equivalence for *symplectic* groupoids and for their infinitesimal counterparts, Poisson manifolds. In Xu's work, two symplectic groupoids are Morita equivalent if there exists a *symplectic* principal bibundle between them. Hence, this is a natural extensions of the first language above to the symplectic setting. However, a glance at the table above raises several natural questions:

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- (Q1) What is a left principal bibundle of symplectic groupoids?
- (Q2) What is a weak equivalence of symplectic groupoids?
- (Q3) What is the fibered category one should associate to a symplectic groupoid?

In this paper we give answers to these questions in terms of both the global objects (groupoids) and the infinitesimal objects (Lie algebroids). This paper is mostly concerned with the former, while the latter will be briefly treated in Section 6.

This paper is organized as follows: In Section 2 we establish our notation and we introduce the site of Dirac manifolds, \mathbf{DMan} . The language of Dirac geometry is convenient since Dirac manifolds are much better behaved categorically than Poisson manifolds. This generalization comes at no cost since Morita equivalence is concerned with the transverse geometry and the non-degeneracy of the leaf-wise 2-forms is irrelevant.

In Section 3 we introduce a new object called a D-Lie groupoid. Briefly, a D-Lie groupoid is just a groupoid internal to \mathbf{DMan} . Intuitively, they can be thought of as ‘pseudo-integrations’ of Dirac structures. We will see that the notion of D-Lie groupoid encompasses many interesting phenomena. For example, Poisson manifolds, symplectic orbifolds, Hamiltonian G spaces, and integrable systems all give rise to natural examples of D-Lie groupoids. In this section we introduce also the infinitesimal version of a D-Lie groupoid: a D-Lie algebroid. This turns out to be a Lie algebroid over a Dirac manifold, equipped with an infinitesimal multiplicative form [3] which is compatible with the Dirac structure.

Symplectic and presymplectic groupoids provide important examples of D-Lie groupoids. In Section 4 we discuss principle D-Lie groupoids bundles and the notions of Morita equivalences, morphisms, and weak equivalences of D-Lie groupoids, will allow us to provide answers to (Q1) and (Q2) above.

In Section 5 we discuss stacks over \mathbf{DMan} . We define a functor \mathbf{B} which relates any D-Lie groupoid to a (presentable) stack over \mathbf{DMan} . Since symplectic groupoids are examples of D-Lie groupoids, this provides an answer to (Q3). Moreover, we prove the following result which provides the bridge with the notion of symplectic Morita equivalence one finds in the literature:

Theorem 1.1. *Let \mathcal{G} and \mathcal{H} be symplectic groupoids. Then \mathcal{G} and \mathcal{H} are symplectically Morita equivalent if and only if any of the following hold:*

- (1) *there exists a principal $(\mathcal{G}, \mathcal{H})$ bibundle of D-Lie groupoids;*
- (2) *there exists a pre-symplectic groupoid \mathcal{G}' and a pair of weak equivalences of D-Lie groupoids $\mathcal{G} \leftarrow \mathcal{G}' \rightarrow \mathcal{H}$;*
- (3) *$\mathbf{B}\mathcal{G}$ is isomorphic to $\mathbf{B}\mathcal{H}$.*

Note that the site \mathbf{DMan} , rather than the site \mathbf{Man} of smooth manifolds plays a fundamental role. Two symplectic groupoids \mathcal{G} and \mathcal{H} may be Morita equivalent as Lie groupoids but *not* symplectically Morita equivalent.

Our results show that stacks over \mathbf{DMan} provide an appropriate categorical formalism for Poisson geometry. For example, the ‘separated’ stacks associated to D-Lie groupoids are closely related to the Poisson manifolds of compact type of Crainic, Fernandes, and Martinez-Torres [9, 10]. Moreover, the obvious forgetful functor $\mathbf{DMan} \rightarrow \mathbf{Man}$ can be canonically extended to a functor of (presentable) stacks over \mathbf{DMan} to (presentable) stacks over \mathbf{Man} . Hence, one may think of a presentable stack over \mathbf{DMan} as a singular object of \mathbf{Man} equipped with additional geometry.

If A is an integrable Lie algebroid we will denote by $\Sigma(A)$ the corresponding source 1-connected Lie groupoid. Recall that any morphism $F : A \rightarrow B$ of integrable Lie algebroids integrates to a Lie groupoid morphism $\mathcal{F} : \Sigma(A) \rightarrow \Sigma(B)$ (see, e.g., [7]). In Section 6 we give the following infinitesimal characterization of weak equivalences:

Theorem 1.2. *Let $F : A \rightarrow B$ be a morphism of integrable Lie algebroids. The corresponding morphism $\mathcal{F} : \Sigma(A) \rightarrow \Sigma(B)$ is a weak equivalence if and only if the following conditions hold for all $x \in M$:*

- (a) F induces a homeomorphism of orbit spaces;
- (b) F is transverse;
- (c) The map of isotropy algebras $F_x : \mathfrak{g}_x \rightarrow \mathfrak{g}_{f(x)}$ is an isomorphism;
- (d) The map of monodromy groups $\mathcal{N}_x(A) \rightarrow \mathcal{N}_x(B)$ is an isomorphism;
- (e) The map of fundamental groups $\pi_1(\mathcal{O}_x) \rightarrow \pi_1(\mathcal{O}_{f(x)})$ is an isomorphism.

This result suggests a definition of infinitesimal weak equivalences of any D-Lie algebroids. By treating a Dirac structure as a D-Lie algebroid, we are able to define a weak equivalence of Dirac manifolds, and this leads to our next theorem:

Theorem 1.3. *Suppose M and N are integrable Poisson manifolds. Let $\Sigma(M)$ and $\Sigma(N)$ be their source simply connected integrations. Then $\Sigma(M)$ and $\Sigma(N)$ are Morita equivalent if and only if there exists a Dirac manifold X and pair of weak equivalences $M \leftarrow X \rightarrow N$.*

While the previous theorem is about integrable Poisson manifolds, the infinitesimal version of weak equivalence is perfectly well defined for non-integrable manifolds. This leads to a natural version of Morita equivalence for non-integrable Poisson manifolds which we call *infinitesimal Morita equivalence*. Two Poisson manifolds M and N are *infinitesimally Morita equivalent* if there exists Dirac manifolds $\{X^i\}$ and a chain of weak equivalences:

$$M \leftarrow X^1 \rightarrow X^2 \leftarrow \dots \rightarrow X^{n-1} \leftarrow X^n \rightarrow N.$$

When M and N are integrable, this coincides with ordinary Morita equivalence of Poisson manifolds.

In [5], Burzstyn, Nosedá, and Zhu characterized principal bundles for *stacky* groupoids, which are the geometric objects that ‘integrate’ non-integrable algebroids. This also gives rise to a natural definition of Morita equivalence of non-integrable algebroids, which we call BNZ equivalence: two algebroids are BNZ equivalent if and only if there exists a principal bibundle of their stacky (holonomy) integrations. We conjecture that infinitesimal Morita equivalence and BNZ equivalence coincide. This will be discussed in a upcoming paper.

Finally, although we only discuss in this paper Dirac structures without a background 3-form, our results should extend without much difficulty to the case where a background 3-form is present.

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2. PRELIMINARIES

Poisson tensors are not well behaved in a category theoretic sense. For example, given a submersion $f : N \rightarrow M$, and a Poisson structure on M , we generally cannot pull-back the Poisson structure on M to N . We can interpret this problem geometrically: If we think of a Poisson manifold, M as a singular foliation by symplectic manifolds, then we can pull back this foliation along any submersion. However, the leaves of the resulting foliation will typically no longer be symplectic manifolds but *presymplectic* manifolds. We can resolve this issue by thinking of Poisson manifolds as special examples of Dirac structures.

2.1. Dirac structures. Intuitively, Dirac manifolds are manifolds equipped with a singular foliation by presymplectic manifolds (i.e. manifolds equipped with a closed 2-form). We will provide a brief overview of Dirac structures in order to establish our notation conventions. For a more detailed discussion on Lie groups see [2] [15]. A brief discussion and motivation of actions and bibundles of Lie groupoids can also be found in Section 2 and 3 of [6].

Given a smooth manifold, M , we call $\mathbb{T}M = TM \oplus T^*M$ the *generalized tangent bundle* of M . Elements of $\mathbb{T}_x M$ will be denoted by pairs $v \oplus \eta$ where $v \in T_x M$ and $\eta \in T_x^* M$. Generally we will use η and ζ to denote 1-forms or co-vectors while α and β will denote 2-forms.

Dirac structures on M is a special subbundles $L \leq \mathbb{T}M$ that come equipped with a natural Lie algebroid structure. We will denote the Lie bracket on sections of L by square brackets $[\cdot]_L$. Furthermore, ρ_L will denote the anchor map $v \oplus \eta \mapsto v$. The orbits (or leaves) of L , are the maximal submanifolds integrating the singular distribution $\rho_L(L)$. The orbits of a Dirac structure come with a closed 2-form denoted $\omega^{\mathcal{O}}$.

We will think of Poisson manifolds as special examples of Dirac manifolds. Given a manifold M equipped with a Poisson bivector π , we will denote the associated Dirac structure by L_π .

2.2. The category of Dirac manifolds. A *Dirac manifold* is a pair (M, L_M) where M is a manifold and L_M is a Dirac structure on M . Given a smooth map $f : N \rightarrow M$ and a structure L_M , we can define a pullback operation:

$$(f^* L_M)_x := \{w \oplus f^* \beta : df(w) \oplus \beta \in (L_M)_x\} \quad (2.1)$$

Geometrically this corresponds to pulling back the foliation on M and the associated leaf-wise 2-forms. Unfortunately this construction does not always result in a proper Dirac structure. The problem is that $f^* L_M$ may not be a smooth vector subbundle. However, if $f : N \rightarrow M$ is transverse to the orbits of L_M then $f^* L_M$ is always a well defined Dirac structure [2]. In particular, Dirac structures can always be pulled back along submersions.

Now suppose (M, L_M) is a Dirac manifold and $\beta \in \Omega^2(M)$ is a closed 2-form on M . Then the *gauge transform* of L_M by β written $L_M + \beta$ is defined as follows:

$$(L_M + \beta)_x := \{(v, \alpha + \iota_v(\beta)) \in \mathbb{T}M_x : (v, \alpha) \in L_M\}. \quad (2.2)$$

Geometrically, this corresponds to adding β to each leaf-wise 2-form.

We can now define the main category of study.

Definition 2.1. The *category of Dirac Manifolds* \mathbf{DMan} is defined as follows:

- the *objects* of \mathbf{DMan} are Dirac manifolds (M, L_M) ;

- the *morphisms* of \mathbf{DMan} are pairs $(f, \beta) : (N, L_N) \rightarrow (M, L_M)$ where f is smooth map such that f^*L_M is a well defined Dirac structure on N and $\beta \in \Omega^2(N)$ is a 2-form such that

$$f^*L_M = L_N + \beta. \quad (2.3)$$

Composition in this category is by the rule $(f, \beta) \circ (g, \alpha) = (f \circ g, g^*\beta + \alpha)$.

Example 2.2 (Gauge Transformations). Suppose (M, L_M) is a Dirac manifold and β is a closed 2-form on M , then $(\text{Id}, \beta) : (M, L_M) \rightarrow (M, L_M + \beta)$ is a morphism in \mathbf{DMan} . We call such morphisms *gauge transformations*.

Example 2.3 (Smooth Maps). Let M and N be any smooth manifolds. Then (M, TM) and (N, TN) are Dirac manifolds. For any smooth map $f : M \rightarrow N$ we have that $(f, 0) : (M, TM) \rightarrow (N, TN)$ is a morphism in \mathbf{DMan} .

Example 2.4 (Symplectic Leaves). Suppose (M, L_M) is a manifold and L_M is the graph of a Poisson bivector. Any orbit \mathcal{O} of M has an associated symplectic form $\omega_{\mathcal{O}}$ and the immersion $i : \mathcal{O} \rightarrow M$ satisfies $i^*L_M = L_{\omega_{\mathcal{O}}}$.

To simplify our notation we will often denote a morphism (f, β) in \mathbf{DMan} by f and the 2-form β will be called the *gauge part* of f . Similarly we may sometimes denote a Dirac manifold (M, L_M) by M alone. The notation L_M will always denote the Dirac structure of M .

The category \mathbf{DMan} comes with a natural functor $\mathbf{Pr}_1 : \mathbf{DMan} \rightarrow \mathbf{Man}$ by projection to the first factor of each pair. This functor is split by a fully faithful functor $\mathbf{i} : \mathbf{Man} \rightarrow \mathbf{DMan}$ which takes any manifold M to the Dirac manifold (M, TM) and any smooth map f to $(f, 0)$.

We can characterize commutative diagrams in \mathbf{DMan} by considering the associated diagram in \mathbf{Man} together with a *gauge equation*. For example, suppose we are given a triangle T of morphisms in \mathbf{DMan} . That is:

$$T = \begin{array}{ccc} M_1 & \xrightarrow{f_3} & M_3 \\ & \searrow f_1 & \nearrow f_2 \\ & M_2 & \end{array} \quad (2.4)$$

The *gauge part* of T is the equation $f_2^*\beta_1 + \beta_2 = \beta_3$ (here β_i is the gauge part of f_i). More generally, any diagram D in \mathbf{DMan} comes with a set of gauge equations coming from each triangle in D . We can see immediately that, D is a commuting diagram if and only if $\text{Pr}_1(D)$ commutes in \mathbf{Man} and each gauge equation holds.

Suppose we are given two morphisms $f : M \rightarrow X$ and $g : N \rightarrow X$ in \mathbf{DMan} such that the manifold $M \times_X N$ exists. Then the *fiber product* is defined to be $M \times_X N$ where we equip $M \times_X N$ with the Dirac structure:

$$L_{M \times_X N} := (f \circ \text{pr}_1)^*L_X - \text{pr}_1^*\beta - \text{pr}_2^*\alpha. \quad (2.5)$$

Such a fiber product fits into a corresponding pullback square in \mathbf{DMan} :

$$\begin{array}{ccc} M \times_X N & \xrightarrow{\text{pr}_2} & N \\ \downarrow \text{pr}_1 & & \downarrow g \\ M & \xrightarrow{f} & X. \end{array}$$

We take the gauge parts of \mathbf{pr}_1 and \mathbf{pr}_2 to be $\mathbf{pr}_2^*\alpha$ and $\mathbf{pr}_1^*\beta$ respectively. Observe that such a fiber product always exists if either $\mathrm{Pr}_1(f)$ or $\mathrm{Pr}_1(g)$ is a submersion in \mathbf{Man} .

Fiber products in \mathbf{DMan} still satisfy the same universal property. Suppose we have the following diagram in \mathbf{DMan} :

$$\begin{array}{ccccc}
 & & & h_2 & \\
 & & & \searrow & \\
 Y & & & & N \\
 & \searrow h_1 & & \downarrow \mathbf{pr}_1 & \downarrow g \\
 & M \times_X N & \xrightarrow{\mathbf{pr}_2} & N & \\
 & \downarrow \mathbf{pr}_1 & & \downarrow g & \\
 & M & \xrightarrow{f} & X &
 \end{array}$$

Let η_1 and η_2 be the gauge parts of h_1 and h_2 , respectively. Then the gauge equation arising from the outermost square is:

$$h_1^*\alpha + \eta_2 = h_2^*\beta + \eta_1. \quad (2.6)$$

We already know that there is a unique smooth map of manifolds $k : Y \rightarrow M \times_X N$ which makes this diagram commute. We can define the gauge part of k , call it κ , one of two ways:

$$\kappa + k^*\mathbf{pr}_1^*\beta = \eta_2 \quad \kappa + k^*\mathbf{pr}_2^*\beta = \eta_1$$

In the presence of (2.6) these definitions are equivalent. They must hold in order for the diagram to commute since they represent the gauge equations of the top and left triangles created by inserting $k : Y \rightarrow M \times_X N$ into the diagram above. Hence, (k, κ) the unique morphism which completes the diagram in \mathbf{DMan} .

2.3. Groupoids and bibundles. We will now briefly establish our notation for Lie groupoids and their principal bibundles. A more detailed exposition on general Lie groupoids can be found in [8] or [15]. For a quick primer on actions and bibundles see Section 3 of [6].

A Lie groupoid is pair of manifolds \mathcal{G} and M together with maps $\mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{m}, \mathbf{i}$ called the source, target, unit, multiplication and inverse, respectively. We require that these maps satisfy the appropriate groupoid axioms and that the source and target maps be submersions. For each natural number $n > 1$ we denote by $\mathcal{G}^{(n)}$ the manifold of n -tuples of composable arrows.

Given Lie groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$, a $(\mathcal{G}, \mathcal{H})$ -bibundle is a manifold P together with maps

$$\mathbf{t}^P : P \rightarrow M, \quad \mathbf{s}^P : P \rightarrow N,$$

$$\mathbf{m}_L : \mathcal{G} \times_M P \rightarrow P, \quad \mathbf{m}_R : P \times_N \mathcal{H} \rightarrow P,$$

satisfying the axioms of commuting left and right actions over N and M , respectively.

P is *left principal* if P/\mathcal{G} is a manifold and the induced map $P/\mathcal{G} \rightarrow N$ is a diffeomorphism. (*right principal* if $P/\mathcal{H} \rightarrow M$ is a diffeomorphism). If P is both left and right principal then P is a principal $(\mathcal{G}, \mathcal{H})$ bibundle. Principal $(\mathcal{G}, \mathcal{H})$ bundles are also known as *Morita equivalences*.

Given a $(\mathcal{G}_1, \mathcal{G}_2)$ -bibundle P_1 and a $(\mathcal{G}_2, \mathcal{G}_3)$ bibundle P_2 , we can construct a tensor product $P_1 \otimes_{\mathcal{G}_2} P_2$ where:

$$P_1 \otimes P_2 := \frac{P_1 \times_{M_2} P_2}{(p_1 \cdot g_2, p_2) \sim (p_1, g_2 \cdot p_2)}.$$

This composition is associative (up to bibundle isomorphism). If P is a $(\mathcal{G}, \mathcal{H})$ bibundle then $\mathcal{G} \otimes P \cong P$ and $P \otimes \mathcal{H} \cong P$. Lastly, P is invertible (i.e. there exists P^{-1} such that $P^{-1} \otimes P \cong \mathcal{H}$ and $P \otimes P^{-1} \cong \mathcal{G}$) if and only if P is principal.

Example 2.5. Given a homomorphism of Lie groupoids $F : \mathcal{H} \rightarrow \mathcal{G}$ covering $f : M \rightarrow N$ then let P_F be the left principal $(\mathcal{G}, \mathcal{H})$ -bibundle constructed as follows: As a manifold $P_F = \mathcal{H} \times_{\mathbf{s}, f} N$. The left and right actions on P_F are given by:

$$\mathbf{m}_L(g', (g, x)) = (g'g, x), \quad \mathbf{m}_R((g, x), h) = (gF(h), \mathbf{s}(h)).$$

The assignment $F \mapsto P_F$ is functorial in the sense that it satisfies

$$P_{F \circ G} \cong P_F \otimes P_G. \quad (2.7)$$

Given a Lie algebroid $(A, [\cdot, \cdot]_A, \rho_A)$ we will use $\Sigma(A)$ to denote the canonical source simply connected integration of A . When $A = L_M$ is a Dirac structure, the notation $\Sigma(L_M)$ denotes the canonical source simply connected pre-symplectic groupoid integrating L_M (see [4] for a treatment of this construction).

3. GROUPOIDS IN \mathbf{DMan}

Definition 3.1. In brief, a *D-Lie groupoid* is a groupoid object internal to the category \mathbf{DMan} . Hence, it is a pair of objects \mathcal{G} and M in \mathbf{DMan} together with morphisms $\mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{m}, \mathbf{i}$ satisfying the groupoid axioms. In general, we also require that $\mathbf{Pr}_1(\mathbf{s})$ and $\mathbf{Pr}_1(\mathbf{t})$ be submersions.

The notion of a D-Lie groupoid should not be confused with the notion of a multiplicative Dirac structure on a groupoid, the so-called Dirac groupoids [13, 18], which include Poisson-Lie groups and Poisson groupoids as special cases. In general, Dirac groupoids *do not* (in any obvious way) provide examples of D-Lie groupoids.

Each groupoid axiom can be interpreted as a diagram in \mathbf{DMan} . Hence, if we are supplied with Dirac manifolds \mathcal{G} and M and morphisms $\mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{m}, \mathbf{i}$ then the resulting data is a D-Lie groupoid if and only if the projection under \mathbf{Pr}_1 is a Lie groupoid and the associated gauge equations of each groupoid axiom holds. A *homomorphism* of D-Lie groupoids is defined in the natural way: It is a pair of morphisms, $F : \mathcal{G} \rightarrow \mathcal{H}$ and $f : M \rightarrow N$, in \mathbf{DMan} , such that F is compatible with the source, target and multiplication maps.

The next lemma allows us to give a more geometric characterization of D-Lie groupoids.

Lemma 3.2. *Let \mathcal{G} be a Lie groupoid over M and L_M be a Dirac structure on M . Suppose τ and σ are a pair of closed 2-forms on \mathcal{G} such that:*

- (i) $\mathbf{t}^* L_M = \mathbf{s}^* L_M + (\tau - \sigma)$;
- (ii) $\tau - \sigma$ is multiplicative.

Then there is a unique D-Lie groupoid whose underlying Lie groupoid is \mathcal{G} and has source and target morphisms (\mathbf{s}, σ) and (\mathbf{t}, τ) . Furthermore, every D-Lie groupoid can be obtained in this manner.

A detailed proof of this lemma can be found in Appendix A. Still, let us outline the main idea.

The key observation is that each groupoid axiom can be interpreted as a commutative diagram. Therefore, each groupoid axiom has an associated gauge equation, which is an equation of 2-forms involving the gauge parts of each structure maps. Examining these equations reveals that the gauge part of each structure map can be written entirely in terms of τ and σ . For instance, gauge part of $\mathbf{s} \circ \mathbf{m} = \mathbf{s} \circ \mathbf{pr}_2$ yields the equation:

$$\mu = \mathbf{pr}_1^* \sigma - \mathbf{m}^* \sigma + \mathbf{pr}_2^* \sigma. \quad (3.1)$$

For the first part of the proof, we first observe $L_{\mathcal{G}} := \mathbf{t}^* L_M - \tau$ is a well defined Dirac structure on \mathcal{G} . Furthermore, condition (i) is equivalent to saying that (\mathbf{t}, τ) and (\mathbf{s}, σ) are morphisms of Dirac manifolds. To construct the D-Lie groupoid, we must produce the 2-forms corresponding to the remaining structure maps. What we do then is take equations such as (3.1) to be definitions. We are left to check that the assumption that $\tau - \omega$ is multiplicative suffices to ensure that this produces a well defined D-Lie groupoid. For the second part, we just need to show that every D-Lie groupoid satisfies (i) and (ii).

From now on we will call the pair (τ, σ) the *gauge pair* of \mathcal{G} . The 2-form $\Omega = \tau - \sigma$ will be called the *characteristic form* of \mathcal{G} . One way to think of condition (i) is that Ω measures the degree to which $\mathbf{t}^* L_M \neq \mathbf{s}^* L_M$. Condition (ii) says that this failure must be up to a multiplicative gauge transformation.

If a D-Lie groupoid \mathcal{G} has a gauge pair of the form $(\tau, 0)$ then $\Omega = \tau$ and we call \mathcal{G} *target aligned*. It turns out that, up to isomorphism, it suffices to consider target aligned D-Lie groupoids.

Lemma 3.3. *Let $\mathcal{G} \rightrightarrows M$ be a D-Lie groupoid over M with gauge pair (τ, σ) . Then the pair $(\tau - \sigma, 0)$ determines a target aligned D-Lie groupoid and the gauge transformation:*

$$(\mathrm{Id}_{\mathcal{G}}, \sigma) : (\mathcal{G}, L_{\mathcal{G}}) \rightarrow (\mathcal{G}, L_{\mathcal{G}} + \sigma),$$

is an isomorphism.

Proof. For ease of notation, let \mathcal{G}' denote the Dirac manifold $(\mathcal{G}, L_{\mathcal{G}} + \sigma)$. By Lemma 3.2, \mathcal{G}' is a D-Lie groupoid with gauge pair $(\tau - \sigma, 0)$. The morphism $(\mathrm{Id}_{\mathcal{G}}, \sigma)$ is clearly an isomorphism of Dirac manifolds. It only remains to check that $(\mathrm{Id}_{\mathcal{G}}, \sigma)$ is a homomorphism of D-Lie groupoids. We first verify that $(\mathrm{Id}_{\mathcal{G}}, \sigma)$ is compatible with the source and target maps, i.e.:

$$(\mathbf{s}, 0) \circ (\mathrm{Id}, \sigma) = (\mathrm{Id}, 0) \circ (\mathbf{s}, \sigma) \quad \text{and} \quad (\mathbf{t}, \Omega) \circ (\mathrm{Id}, \sigma) = (\mathrm{Id}, 0) \circ (\mathbf{t}, \tau). \quad (3.2)$$

The above equalities are clear from the definition of composition in \mathbf{DMan} . To see that $(\mathrm{Id}_{\mathcal{G}}, \sigma)$ is compatible with the multiplication, we must check that:

$$\begin{array}{ccc} \mathcal{G} \times_M \mathcal{G} & \xrightarrow{(\mathrm{Id}, \mathbf{pr}_1^* \sigma + \mathbf{pr}_2^* \sigma)} & \mathcal{G}' \times_M \mathcal{G}' \\ \downarrow (\mathbf{m}, \mu) & & \downarrow (\mathbf{m}, 0) \\ \mathcal{G} & \xrightarrow{(\mathrm{Id}, \sigma)} & \mathcal{G}' \end{array}$$

commutes. The gauge equation associated to this diagram is $\mathbf{m}^* \sigma + \mu = \mathbf{pr}_1^* \sigma + \mathbf{pr}_2^* \sigma$ which holds by (3.1). \square

Next we give two illustrating examples of D-Lie groupoids which characterize known phenomena.

Example 3.4 (Symplectic Groupoids). Given (\mathcal{G}, Ω) , a symplectic groupoid integrating (M, L_M) , then $t^*L_M = s^*L_M + \Omega$. Therefore, a symplectic groupoid is the same as a target aligned D-Lie groupoid with non-degenerate characteristic form.

Example 3.5 (Symplectic orbifolds). Let $\mathcal{G} \rightrightarrows$ be an étale Lie groupoid and suppose ω is a symplectic form on M . Suppose further that $\mathbf{t}^*\omega - \mathbf{s}^*\omega = 0$. \mathcal{G} can be thought of as the presentation of a (possibly non-effective) symplectic orbifold. By thinking of M as a Dirac manifold, then \mathcal{G} can also be thought of as a D-Lie groupoid with characteristic form 0.

The two preceding can be thought of as extreme cases of D-Lie groupoids. In the first case, the characteristic 2-form on the space of arrows is non-degenerate. In the second case, the characteristic 2-form vanishes. In general, a typical D-Lie groupoid is something in between a symplectic groupoid and a groupoid equipped with a multiplicative Dirac structure on the space of objects.

3.1. D-Lie groupoid morphisms. We will now take a closer look at homomorphisms of D-Lie groupoids. Throughout, \mathcal{G} and \mathcal{H} are D-Lie groupoids over M and N respectively. Also, $\Omega^{\mathcal{G}}$ and $\Omega^{\mathcal{H}}$ will denote their respective characteristic forms.

Continuing from our work on Lemma 3.2, we can also interpret morphisms of D-Lie groupoids in terms of the characteristic forms.

Lemma 3.6. *Suppose $F : \mathcal{G} \rightarrow \mathcal{H}$ is a homomorphism of the underlying Lie groupoids covering the smooth map $f : M \rightarrow N$. Let β be a closed 2-form such that (f, β) is a morphism of Dirac manifolds and:*

$$F^*\Omega^{\mathcal{H}} = t^*\beta - s^*\beta + \Omega^{\mathcal{G}}. \quad (3.3)$$

Then there is a unique 2-form α making (F, α) into a D-Lie groupoid morphism covering (f, β) . Furthermore, every D-Lie groupoid morphism is of this sort.

Proof. To prove the first part, we just need to supply a suitable α . Let $(\tau^{\mathcal{G}}, \sigma^{\mathcal{G}})$ and $(\tau^{\mathcal{H}}, \sigma^{\mathcal{H}})$ be the gauge pairs of \mathcal{G} and \mathcal{H} . Now define α to be the unique 2-form satisfying:

$$F^*\sigma^{\mathcal{H}} + \alpha = s^*\beta + \sigma^{\mathcal{G}}. \quad (3.4)$$

If α does the job, it is certainly the unique one since (3.4) is the gauge part of compatibility with the source. The gauge part of compatibility with the target is the analogous equation:

$$F^*\tau^{\mathcal{H}} + \alpha = s^*\beta + \tau^{\mathcal{G}}. \quad (3.5)$$

Which follows from combining (3.3) and (3.4). We leave it to the reader to verify that the gauge part of compatibility with multiplication follows from (3.4) and (3.5). Hence (F, α) is a morphism of D-Lie groupoids.

Certainly all morphism of D-Lie groupoids will be of this form since (3.3) can be obtained by subtracting (3.4) from (3.5). \square

Example 3.7 (Symplectic Groupoids). Suppose $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ are symplectic groupoids. If we think of \mathcal{G} and \mathcal{H} as target aligned D-Lie groupoids then a morphism consists of a homomorphism of Lie groupoids $F : \mathcal{G} \rightarrow \mathcal{H}$ together with a closed 2-form $\beta \in \Omega^2(M)$ such that (3.3) holds.

3.2. D-Lie algebroids. There is an infinitesimal version of D-Lie groupoids. Recall from [3] that given a closed multiplicative form on a Lie groupoid, there is a corresponding *infinitesimal multiplicative form*. In the case of closed 2-forms an infinitesimal multiplicative form is a bundle map:

$$\rho_A^* : A \rightarrow T^*M.$$

Here, A is the Lie algebroid of the Lie groupoid in consideration. We will now give a brief construction of ρ_A^* .

Lemma 3.8. *The bundle map:*

$$\rho_A^* : A \rightarrow T^*M \quad v \mapsto \alpha : t^*\alpha = \Omega^b(v),$$

is well defined.

Proof. Suppose $v \in A_x := \ker(ds)|_{\mathbf{u}(x)}$. Since, $v \in \ker ds$ we have that $v \oplus 0 \in s^*L_M$. Therefore:

$$v \oplus \Omega^b(v) \in s^*L_M + \Omega = t^*L_M.$$

Hence there exists a unique $\alpha \in T_x^*M$ such that $t^*\alpha = \Omega^b(v)$ and so $dt(v) \oplus \alpha \in L_M$. \square

It turns out that ρ^* is compatible with L_M in the following way:

$$\rho(v) \oplus \rho^*(v) \in L_M \leq TM \oplus T^*M, \quad \forall v \in A. \quad (3.6)$$

Definition 3.9. A *D-Lie Algebroid* is a Lie algebroid $(A, [\cdot, \cdot]_A, \rho)$ over a manifold M together with an IM 2-form $\rho^* : A \rightarrow T^*M$ and a Dirac structure L_M on M such that (3.6) holds.

If \mathcal{G} is a D-Lie groupoid over (M, L_M) with characteristic form Ω , then let A be the algebroid of the Lie groupoid $\text{Pr}_1(\mathcal{G})$ and let ρ_A^* be defined as above. In such a case we say that \mathcal{G} integrates (A, ρ_A^*) . When \mathcal{G} is source simply connected and target aligned we say that \mathcal{G} is the canonical integration.

Proposition 3.10. *A D-Lie algebroid is integrable if and only if A is integrable.*

Proof. To see this, first observe that the main theorem of [3] says that infinitesimal multiplicative 2-forms on A are in 1-1 correspondence with multiplicative 2-forms Ω on the source simply connected integration $\Sigma(A)$. From there, it only remains to check that $\mathbf{t}^*L_M = s^*L_M + \Omega$ if and only if (3.6) holds.

Suppose $v \in A$ and let $\alpha = \rho_A^*(v)$. Assume that $\mathbf{t}^*L_M = s^*L_M + \Omega$. Since $v \in \ker ds$ we know that $v \oplus 0 \in s^*L_M$. Using our assumption we can conclude that $v \oplus \Omega^b(v) \in \mathbf{t}^*L_M$. By the definition of ρ_A^* , $v \oplus \mathbf{t}^*\alpha \in \mathbf{t}^*L_M$ and so $dt(v) \oplus \alpha \in L_M$. Hence we have shown that $\rho_A(v) \oplus \rho_A^*(v) \in L_M$.

Now assume that (3.6) holds. We now need to show that $\mathbf{t}^*L_M = s^*L_M + \Omega$. We will show that these vector bundles are identical over the image of the unit map $\mathbf{u}(M)$ and leave it to the reader that this implies it must hold everywhere. Recall that Dirac structures are maximally isotropic with respect to the following bilinear symmetric product on TM :

$$\langle v \oplus \eta, w \oplus \zeta \rangle = \zeta(v) + \eta(w).$$

Hence, it suffices to show that the product of an element of \mathbf{t}^*L_M with an element of $s^*L_M + \Omega$ vanishes. Let $v \in T_M\mathcal{G}$. Since the unit map splits the source map we can write $v = v_A + v_u$ where $v_A \in A$ and $v_u \in \mathbf{du}(TM)$. Since we have assumed

that (3.6) holds, it follows that $v_A \oplus \Omega^b(v_A) \in \mathfrak{t}^*L_M$. Therefore, we can split \mathfrak{t}^*L_M into sums of the form:

$$(v_A \oplus \Omega^b(v_A)) + (v_u \oplus \mathfrak{t}^*\eta).$$

On the other hand, given any $w_A \in A$ we know that $w_A \oplus 0 \in \mathfrak{s}^*L_M$. Therefore $w_A \oplus \Omega^b(w_A) \in \mathfrak{s}^*L_M + \Omega$. Therefore, we can split $\mathfrak{s}^*L_M + \Omega$ in a similar manner:

$$(w_A \oplus \Omega^b(w_A)) + (w_u \oplus \mathfrak{s}^*\zeta + \Omega^b(w_u)).$$

Since we have shown that elements of the form $A \oplus \Omega^b(A)$ are common to both, we only need to check that:

$$\langle v_u \oplus \mathfrak{t}^*\eta, w_u \oplus \mathfrak{s}^*\zeta + \Omega^b(w_u) \rangle = 0.$$

The left hand side simplifies to:

$$\mathfrak{t}^*\eta(w_u) + \mathfrak{s}^*(v_u) = \mathfrak{s}^*\eta(w_u) + \mathfrak{s}^*\zeta(v_u) = \langle v_u \oplus \mathfrak{s}^*\eta, w_u \oplus \mathfrak{s}^*\zeta \rangle.$$

Notice that the right most expression is actually the product of two elements of \mathfrak{s}^*L_M and hence must vanish. \square

Definition 3.11. Suppose (A, ρ_A^*) and (B, ρ_B^*) are D-Lie algebroids. A *morphism* of D-Lie algebroids is a Lie algebroid morphism $F : A \rightarrow B$ which cover a morphism of Dirac manifolds $(f, \beta) : M \rightarrow N$ such that

$$f^* \circ \rho_B^* \circ F = \rho_A^* + \beta^b$$

The left side of the equation above is the *pullback* of the IM 2-form ρ_B^* along F . The right side is the infinitesimal form of the gauge transformation $\Omega + \mathfrak{t}^*\beta - \mathfrak{s}^*\beta$.

4. PRINCIPAL BUNDLES AND MORITA EQUIVALENCE

In this section, we will define G bundles and $(\mathcal{G}, \mathcal{H})$ -bibundles in the setting of D-Lie groupoids. This gives rise to a notion of Morita equivalence of D-Lie groupoids which generalizes Morita equivalence for symplectic groupoids.

4.1. Principal \mathcal{G} bundles.

Definition 4.1. Suppose $\mathcal{G} \rightrightarrows M$ is a D-Lie groupoid. A *left \mathcal{G} bundle* over N is defined to be manifold P equipped with morphisms $\mathfrak{s}^P : P \rightarrow N$, $\mathfrak{t}^P : P \rightarrow M$ and $\mathfrak{m}_L : \mathcal{G} \times_M P \rightarrow P$ satisfying the axioms of a left \mathcal{G} action over N . We say P is principal if the underlying smooth maps of manifolds form a principal bundle.

The morphisms $\mathfrak{s}^P, \mathfrak{t}^P$ and \mathfrak{m}_L^P come with gauge parts σ^P, τ^P and μ_L^P . The equation associated to $\mathfrak{s}^P \circ \mathfrak{m}_L(g, p) = \mathfrak{s}^P(p)$ is:

$$\mu_L + \mathfrak{m}^*\sigma^P = \text{pr}_1^*\sigma^{\mathcal{G}} + \text{pr}_2^*\sigma^P.$$

Similarly gauge equation of $\mathfrak{t}^P \circ \mathfrak{m}_L(g, p) = \mathfrak{t}^P(g)$ is:

$$\mu_L + \mathfrak{m}^*\tau^P = \text{pr}_1^*\tau^{\mathcal{G}} + \text{pr}_2^*\tau^P.$$

Therefore, when defining a principal \mathcal{G} bundle it suffices to specify the 2-forms σ^P and τ^P . As with D-Lie groupoids, we say the characteristic 2-form of P is $\Omega^P := \tau^P - \sigma^P$. Let $\Omega^{\mathcal{G}}$ be the characteristic 2-form of \mathcal{G} . Then:

$$\mathfrak{m}_L^*\Omega^P = \text{pr}_1^*\Omega^{\mathcal{G}} + \text{pr}_2^*\Omega^P,$$

i.e. the closed 2-form Ω^P is left multiplicative. When $\sigma^P = 0$ we call P *target aligned*. By a similar argument as in Lemma 3.3, every principal \mathcal{G} -bundle is canonically isomorphic to a target aligned principal \mathcal{G} bundle.

The next three lemmas are important technical results about principal \mathcal{G} bundles that will be needed later in Section 5. In brief, the first says that the standard construction of a pullback \mathcal{G} bundle still works in our setting. The second lemma implies that the pullback construction is unique up to a unique isomorphism. The last lemma says that principal \mathcal{G} bundles of D-Lie groupoids satisfy a property known as ‘descent.’

Lemma 4.2. *Suppose $f : N \rightarrow M$ is a morphism in \mathbf{DMan} and P is a principal \mathcal{G} bundle over M for \mathcal{G} a D-Lie groupoid. Then there exists a principal \mathcal{G} bundle Q and an principal bundle morphism $F : Q \rightarrow P$ covering f .*

Proof. Without loss of generality, we can assume \mathcal{G} and P are target aligned. We already know that the result holds in \mathbf{Man} for principal \mathcal{G} bundles of Lie groupoids. Let:

$$Q := f^*P = P \times_{N_1} N_2,$$

and equip it with the standard structure maps so that Q is a principal \mathcal{G} bundle for the underlying Lie groupoid. Let $F : Q \rightarrow P$ be projection to the first coordinate. To make Q a principal \mathcal{G} bundle in \mathbf{DMan} we must equip it with a characteristic form. Let:

$$\Omega^Q := \mathbf{pr}_1^* \Omega^P + \mathbf{pr}_2^* \beta, .$$

It can easily be verified that Ω^Q is left multiplicative with respect to the left action of \mathcal{G} on Q . Furthermore the map $F := \mathbf{pr}_1 : Q \rightarrow P$ is equivariant with respect to the left action. \square

Lemma 4.3. *Suppose $(f_1, \beta_1) : N_1 \rightarrow N_2$ and $(f_2, \beta_2) : N_2 \rightarrow N_3$ are morphisms in \mathbf{DMan} and P_1, P_2 and P_3 are principal \mathcal{G} bundles over N_1, N_2 and N_3 respectively. Furthermore, assume we are given principal bundle morphism $F : P_2 \rightarrow P_3$ and $G : P_1 \rightarrow P_3$ covering f_2 and $f_2 \circ f_1$ respectively. Then there exists a unique principal bundle morphism $F' : P_1 \rightarrow P_2$ such that $F \circ F' = G$.*

Proof. At the level of manifolds, this is a well known property of principal \mathcal{G} bundles for \mathcal{G} a Lie group. Therefore, to define F' it suffices to provide the gauge part of F' . Therefore, let the gauge part of F' be defined to be:

$$\tilde{\beta} := \sigma^{P_1} - (\mathbf{s}^{P_1})^* \beta_1 - (F')^* \sigma^{P_2}.$$

We leave it to the reader to check that $(F, \tilde{\beta})$ is a well defined morphism of principal bundles. Finally, compatibility with the source maps implies that this choice of gauge part is the only one possible and therefore $(F', \tilde{\beta})$ is unique. \square

We will typically denote the principal bundle Q from Lemma 4.2 with the notation f^*P and call it the *pullback bundle* along f . When $f : N \rightarrow M$ is an inclusion, we may also denote f^*P by $P|_N$.

Lemma 4.4. (a) *Suppose $\{i_a : U_a \rightarrow M\}$ is a covering of M in \mathbf{DMan} and \mathcal{G} is a D-Lie groupoid. Let $P_a \rightarrow U_a$ be a collection of principal \mathcal{G} bundles together with morphisms $\phi_{ab} : P_b|_{U_{ab}} \rightarrow P_a|_{U_{ab}}$ such that $\phi_{ab} \circ \phi_{bc} = \phi_{ac}$ when restricted to any triple intersection U_{abc} . Then there exists a principal \mathcal{G} bundle $P \rightarrow M$ together with morphisms $\{\phi_a : P|_{U_a} \rightarrow P_a\}_{a \in A}$ such that $\phi_{ab} \circ \phi_b = \phi_a$.*

(b) Let $P \rightarrow M$ and $Q \rightarrow M$ be principal \mathcal{G} -bundles over M and suppose $\{i_a : U_a \rightarrow M\}$ is a covering of M in \mathbf{DMan} . Suppose we have a collection of morphisms $F_a : P|_{U_a} \rightarrow Q|_{U_a}$ covering the identity such that:

$$F_a|_{P|_{U_{ab}}} = F_b|_{Q|_{U_{ab}}},$$

then there exists a unique morphism $F : P \rightarrow Q$ such that $F|_{P|_{U_a}} = F_a$.

Proof. We first prove (a). As with the previous two lemmas, the result already holds in the smooth category. Let the manifold P and the smooth maps ϕ_a be ones satisfying these properties in the smooth setting. Without loss of generality we can assume each P_a is target aligned. We can also assume without loss of generality that the gauge part of each $\iota_a : U_a \rightarrow M$ is zero. To make P into a target aligned principal bundle in \mathbf{DMan} , we must specify its characteristic form Ω :

$$\Omega^P|_{P|_{U_a}} := \phi_a^* \Omega^{P_a}.$$

This is well defined since:

$$\phi_a^* \Omega^{P_a}|_{P|_{U_{ab}}} = \phi_b^* \Omega^{P_b}|_{P|_{U_{ab}}}.$$

Which follows from the fact that $\phi_{ab}^* \Omega^{P_b} = \Omega^{P_a}$ together with the fact that the smooth maps $\phi_{ab} \circ \phi_b$ and ϕ_a are equal.

We now prove part (b). Again, assume the maps $i_a : U_a \rightarrow M$ have trivial gauge part and P and Q are target aligned. As with part (a) the result is known to hold in the smooth category. The smooth map $F : P \rightarrow Q$ covers the identity, so we only need to show that $F^* \Omega^Q = \Omega^P$. However, we know that $F_a^* \Omega^Q|_{P|_{U_a}} = \Omega^P|_{P|_{U_a}}$ so the claim follows immediately. \square

4.2. Bibundles. We can now proceed to the topic of bibundles and Morita equivalence. Throughout this section \mathcal{G} and \mathcal{H} are D-Lie groupoids over the Dirac manifolds M and N respectively.

Definition 4.5. Suppose \mathcal{G} and \mathcal{H} are D-Lie groupoids. A $(\mathcal{G}, \mathcal{H})$ *bibundle* is defined to be a bibundle object internal to the category \mathbf{DMan} . Hence, it is an object P in \mathbf{DMan} together with morphisms $\mathbf{t}^P, \mathbf{s}^P, \mathbf{m}_L, \mathbf{m}_R$ (again in \mathbf{DMan}) which satisfy the axioms of commuting left and right actions over N and M , respectively.

A bibundle P is said to be *(left/right) principal*, or *principal*, if the underlying bibundle of Lie groupoids (left/right) principal or principal respectively. A principal $(\mathcal{G}, \mathcal{H})$ -bibundle is also called a *Morita equivalence* of \mathcal{G} and \mathcal{H} .

Just like \mathcal{G} -bundles, a $(\mathcal{G}, \mathcal{H})$ -bibundle P of D-Lie groupoids is determined by the data of the underlying bundle and the gauge part of the source and target maps σ^P and τ^P . The *characteristic form* Ω^P of P is defined to be $\sigma^P - \tau^P$ as before. Using the same techniques as before we can show that σ^P and τ^P define a bibundle if and only if Ω^P is left and right multiplicative.

$$\mathbf{m}_L^* \Omega^P = \mathbf{pr}_1^* \Omega + \mathbf{pr}_2^* \Omega^P \quad \mathbf{m}_R^* \Omega^P = \mathbf{pr}_1^* \Omega^P + \mathbf{pr}_2^* \Omega^{\mathcal{H}}.$$

We say that P is *target aligned* if $\sigma^P = 0$.

An equivariant map of $(\mathcal{G}, \mathcal{H})$ bibundles is a morphism $\Phi : P \rightarrow Q$ which commutes with the source and target maps and respects the multiplication. In terms of the characteristic 2-form the condition on $F : Q \rightarrow P$ is just:

$$F^* \Omega^P = \Omega^Q.$$

This makes sense when compared to the case of left \mathcal{G} -bundles since we can think of any $(\mathcal{G}, \mathcal{H})$ -bibundle morphism as a left \mathcal{G} -bundle morphism covering the identity on N .

The next few examples demonstrate how this notion of Morita equivalence of D-Lie groupoid relates to existing definitions of Morita equivalence.

Example 4.6 (Morita Equivalence of Lie groupoids). Given a Morita equivalence P of Lie groupoids \mathcal{G} and \mathcal{H} , then thinking of \mathcal{G} and \mathcal{H} as D-Lie groupoids with the tangent Dirac structure allows us to view (P, TP) as a Morita equivalence of D-Lie groupoids $(\mathcal{G}, T\mathcal{G})$ and $(\mathcal{H}, T\mathcal{H})$. Furthermore, it is a simple exercise to check that any Morita equivalence of the D-Lie groupoids $(\mathcal{G}, T\mathcal{G})$ and $(\mathcal{H}, T\mathcal{H})$ is isomorphic to such a (P, TP) .

Example 4.7 (Symplectic Morita Equivalence). Given a symplectic Morita equivalence (P, Ω^P) of symplectic groupoids $(\mathcal{G}, \Omega^{\mathcal{G}})$ and $(\mathcal{H}, \Omega^{\mathcal{H}})$, we can think of (P, Ω^P) as a target aligned Morita equivalence of \mathcal{G} and \mathcal{H} viewed as D-Lie groupoids.

We can improve on the observation from the preceding example. In fact:

Proposition 4.8. *Suppose \mathcal{G} and \mathcal{H} are target aligned D-Lie groupoids with symplectic characteristic forms. There is a 1-1 correspondence between symplectic Morita equivalences and principal $(\mathcal{G}, \mathcal{H})$ bibundles.*

Proof. Suppose P is a Morita equivalence of Lie groupoids \mathcal{G} and \mathcal{H} and $\Omega^{\mathcal{G}}$ and $\Omega^{\mathcal{H}}$ are 2-forms making $(\mathcal{G}, \Omega^{\mathcal{G}})$ and $(\mathcal{H}, \Omega^{\mathcal{H}})$ into symplectic groupoids. Then we must show that if Ω^P is a left and right multiplicative closed 2-form on P then Ω^P is symplectic.

Let $p \in P$ be a fixed point such that $\mathbf{s}^P(p) = x \in N$ and $\mathbf{t}(p) = y \in M$. Suppose $e : U \rightarrow P$ is a local section around x such that $e(x) = p$. Finally we define $f := \mathbf{s}^P \circ e$ and notice that $f^*L_M = L_N|_U + \beta$ where $\beta = u^*\Omega^P$. Locally, we can model P with a fiber product:

$$P|_U := \mathcal{G} \times_M U.$$

Therefore the tangent space $T_p P$ can be written as pairs $(v, w) \in T_{1_{f(x)}} \mathcal{G} \times T_x N$, such that $\mathbf{ds}(v) = \mathbf{df}(w)$. By multiplicativity:

$$\Omega^P((v_1, w_1), (v_2, w_2)) = \Omega^{\mathcal{G}}(v_1, v_2) + \beta(w_1, w_2).$$

Now suppose that $v_1 \oplus w_1$ is in there kernel of Ω^P . Then:

$$0 = \Omega^P((v_1, v_2), (w_1, 0)) = \Omega^{\mathcal{G}}(v_1, v_2).$$

Therefore $v_1 \in (\ker \mathbf{ds})^\perp = \ker \mathbf{dt}$. Now suppose $(w_1, w_2) = (\mathbf{du} \mathbf{df}(w_2), w_2)$ and assume $w_2 \in T_x \mathcal{O}_x$. Then:

$$\begin{aligned} 0 &= \Omega^{\mathcal{G}}(v_1, \mathbf{du} \mathbf{df}(w_2)) + \beta(v_2, w_2) \\ 0 &= \omega_{\mathcal{O}_y}(\mathbf{df}(v_2), \mathbf{df}(w_2)) + \beta(v_2, w_2) \\ 0 &= -f^* \omega_{\mathcal{O}_y}(v_2, w_2) + \beta(v_2, w_2) \\ f^* \omega_{\mathcal{O}_y}(v_2, w_2) &= \beta(v_2, w_2) \end{aligned}$$

On the other hand, since $f^*L_M = L_N + \beta$ we have that:

$$f^* \omega_{\mathcal{O}_y} = \omega_{\mathcal{O}_x} + \beta.$$

Therefore $\beta(v_2, w_2) = 0$ for all $w_2 \in T_x \mathcal{O}_x$. Since $\omega(\mathcal{O}_x)$ is a symplectic form, we conclude that $v_2 = 0$. We must show that $v_1 = 0$, to do this we will prove that

$v_1 \in \ker \Omega^{\mathcal{G}}$. Let $w \in T_{1_y} \mathcal{G}$ be any tangent vector. Then we can uniquely write w in the form $w_A + w_{\mathbf{u}}$ for $w_A \in \ker ds$ and $w_{\mathbf{u}} \in T_y M$. So:

$$\Omega^{\mathcal{G}}(v_1, w) = \Omega^{\mathcal{G}}(v_1, w_A) + \Omega^{\mathcal{G}}(v_1, w_{\mathbf{u}}) = 0 + \Omega^{\mathcal{G}}(v_1, w_{\mathbf{u}}).$$

Since f is transverse to the foliation on M , we can write $w_{\mathbf{u}} = df(w_f) + w_{\mathcal{O}}$, where w_f is in the image of $d_x f$ and $w_{\mathcal{O}} \in T_y \mathcal{O}_y$. Then:

$$\Omega^{\mathcal{G}}(v_1, w_{\mathbf{u}}) = \Omega^{\mathcal{G}}(v_1, df(w_f)) + \Omega^{\mathcal{G}}(v_1, w_{\mathcal{O}}).$$

Observe that:

$$\Omega^{\mathcal{G}}(v_1, w_f) = \Omega^{\mathcal{G}}(v_1, d\mathbf{u} df(w_f)) + \beta(0, w_f) = 0,$$

and:

$$\Omega^{\mathcal{G}}(v_1, w_{\mathcal{O}}) = \omega_{\mathcal{O}_x}(d\mathbf{t}(v_1), w_{\mathcal{O}}) = 0.$$

Therefore $v_1 = 0$. So we have shown that $\ker \Omega^P$ is trivial so Ω^P is a symplectic form. \square

There is a similar result for pre-symplectic groupoids, which we omit for brevity.

4.3. Weak equivalences. Let \mathcal{G} and \mathcal{H} be D-Lie groupoids over M and N . Suppose $F : \mathcal{H} \rightarrow \mathcal{G}$ is a morphism of D-Lie groupoids covering $f : N \rightarrow M$. Then we can construct a left principal $(\mathcal{G}, \mathcal{H})$ -bibundle as follows:

$$P_F := \mathcal{G} \times_{\mathbf{s}, f} N.$$

The left action is taken to be the map:

$$\mathbf{m}_L := \mathbf{m}(\mathbf{pr}_1 \times \mathbf{pr}_2) \times \mathbf{pr}_3 : \mathcal{G} \times_{\mathbf{s}, \mathbf{t}} \mathcal{G} \times_{\mathbf{s}, f} N \rightarrow \mathcal{G} \times_{\mathbf{s}, f} N$$

And the right action:

$$\mathbf{m}_R := \mathbf{m}(\mathbf{pr}_1 \times (F \circ \mathbf{pr}_3)) \times \mathbf{s} \circ \mathbf{pr}_3 : \mathcal{G} \times_{\mathbf{s}, f} N \times_{\mathbf{pr}_2, \mathbf{t}} \mathcal{H} \rightarrow \mathcal{G} \times_{\mathbf{s}, f} N$$

This is the same as the standard construction for Lie groupoids. The reader should check that these actions satisfy the axioms of a $(\mathcal{G}, \mathcal{H})$ -bibundle. The characteristic form of P_F is:

$$\Omega^{P_F} = \mathbf{pr}_1^* \Omega^{\mathcal{G}} + \mathbf{pr}_2^* \beta,$$

where β is the gauge part of $f : N \rightarrow M$.

We say that F is a *weak equivalence* if P_F is a principal $(\mathcal{G}, \mathcal{H})$ -bibundle. Recall that, by definition, P_F is principal if and only if it is a principal bibundle of Lie groupoids. Therefore, F is a weak equivalence if and only if the underlying map of Lie groupoids is a weak equivalence. This immediately gives rise to a notion of symplectic weak equivalences.

Example 4.9 (Symplectic Weak Equivalences). Suppose \mathcal{G} and \mathcal{H} are symplectic groupoids. I.e., \mathcal{G} and \mathcal{H} are target aligned D-Lie groupoids and their characteristic 2-forms $\Omega^{\mathcal{G}}$ and $\Omega^{\mathcal{H}}$ are symplectic. Then a weak equivalence $F : \mathcal{H} \rightarrow \mathcal{G}$, is a homomorphism of Lie groupoids, together with a closed 2-form β on N such that the following properties hold:

- (a) $F : \mathcal{H} \rightarrow \mathcal{G}$ is fully faithful and essentially surjective;
- (b) $f : N \rightarrow M$ is transverse to π_M (the Poisson structure on M);
- (c) $F^* \Omega^{\mathcal{G}} = \Omega^{\mathcal{H}} + \mathbf{t}^* \beta - \mathbf{s}^* \beta$.

Composition of homomorphisms corresponds to the tensor product operation at the level of bimodules. Given a left principal $(\mathcal{G}_1, \mathcal{G}_2)$ -bibundle P and a left principal $(\mathcal{G}_2, \mathcal{G}_1)$ -bibundle Q . Assume that $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$, P and Q are all target aligned. Thinking of the \mathcal{G}_i as Lie groupoids then $P \otimes Q$ is defined to be:

$$P \otimes Q := P \times_{M_2} Q / \mathcal{G}_2$$

where the action of \mathcal{G}_2 on (p, q) is defined to be $g_2 \cdot (p, q) = (p \cdot g_2^{-1}, g \cdot q)$. In order to equip $P \otimes Q$ into a target aligned left principal $(\mathcal{G}_1, \mathcal{G}_3)$ -bibundle, we only need to equip it with a multiplicative 2-form $\Omega^{P \otimes Q}$. Multiplicativity of Ω^P and Ω^Q with respect to the action of \mathcal{G}_2 ensures that:

$$\tilde{\Omega} := \text{pr}_1^* \Omega^P + \text{pr}_2^* \Omega^Q,$$

is basic with respect to the action of \mathcal{G}_2 on $P \times_{M_2} Q$. Hence, $\tilde{\Omega}$ descends to a 2-form on $\Omega^P \otimes \Omega^Q$. Left and right multiplicativity of $\Omega^{P \otimes Q}$ can easily be checked.

The reader is now encouraged to check the following standard facts:

- There is a (weak) 2-category whose objects are D-Lie groupoids, 1-morphisms are left principal bibundles, and whose 2-morphisms are isomorphisms of bibundles.
- The mapping $F \mapsto P_F$ is functorial (i.e. $P_{F \circ G} \cong P_F \otimes P_G$).
- P_F has a (weak) inverse if and only if F is a weak equivalence.

5. STACKS OVER \mathbf{DMan}

As we noted in our introduction, there are a variety of languages which can be used to study stacks. Our definitions and notations for *category fibered in groupoids* (CFG), *stacks*, *representable* stacks and morphisms, monomorphisms and epimorphisms of CFGs are identical to those found in [5]. For more complete references on fibered categories and stacks we refer the reader to [14] [1] or [19]. The main difference here is that rather than taking these objects to be over the site of smooth manifolds, we take these all to be objects over the site \mathbf{DMan} . Hence, a CFGs and stacks over \mathbf{DMan} should consist of a category \mathcal{X} and a functor $\pi : \mathcal{X} \rightarrow \mathbf{DMan}$ satisfying the appropriate axioms. We will now give some important examples of stacks over \mathbf{DMan} .

Example 5.1 (Dirac Manifolds). As usual, objects in \mathbf{DMan} give rise to CFGs. Suppose M is a Dirac manifold, let \underline{M} be the category where the objects are morphisms $f : N \rightarrow M$ and arrows are commutative triangles:

$$\begin{array}{ccc} N & \xrightarrow{g} & O \\ & \searrow f & \swarrow h \\ & M & \end{array} \quad \mapsto_{\pi} \quad N \xrightarrow{g} O$$

Then \underline{M} is a CFG. In fact, it can be easily check that \underline{M} satisfies the axioms of a stack over \mathbf{DMan} .

Just as in the case of manifolds, this assignment is functorial. In fact, given a Dirac map $f : M \rightarrow N$ then there is a obvious morphism $\underline{f} : \underline{M} \rightarrow \underline{N}$. Furthermore, any morphism $\mathcal{F} : \underline{M} \rightarrow \underline{N}$ is isomorphic to a unique map \underline{f} . Finally, we note that given a morphism $f : M \rightarrow N$ the associated $\underline{f} : \underline{M} \rightarrow \underline{N}$ is an epimorphism if and only if the underlying smooth map of manifolds is a submersion.

Example 5.2 (Principal Bundles). Suppose \mathcal{G} is a D-Lie groupoid. Let $\mathbf{B}\mathcal{G}$ denote the category whose objects are left principal \mathcal{G} -bundles and morphisms are equivariant maps $F : P \rightarrow Q$. Let the functor $\pi : \mathbf{B}\mathcal{G} \rightarrow \mathbf{DMan}$ send $F : P \rightarrow Q$ to $f : M \rightarrow N$. That $\mathbf{B}\mathcal{G}$ satisfies the axioms of a CFG is the content of Lemma 4.2 and Lemma 4.3. Furthermore, $\mathbf{B}\mathcal{G}$ satisfies descent by Lemma 4.4.

Example 5.3 (Poisson Manifolds of Proper Type). Suppose M is a Poisson manifold and \mathcal{G} is a proper symplectic groupoid integrating M . Then $\mathbf{B}\mathcal{G}$ is a *separated stack*. Suppose \mathcal{X} is a stack over \mathbf{DMan} and let $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ be the diagonal embedding. Then we say \mathcal{X} is *separated* if Δ is a proper map. When \mathcal{X} admits a presentation, then this condition is equivalent to the existence of a proper D-Lie groupoid $\mathcal{G} \rightrightarrows M$ such that $\mathbf{B}\mathcal{G} \cong \mathcal{X}$. If we further require that \mathcal{G} is a symplectic groupoid, then M is a Poisson manifold of proper type per Crainic, Fernandes and Martinez Torrez [10] [9].

5.1. Presentations and groupoids. Suppose \mathcal{X} is a stack and \underline{M} is a representable CFG. Then a representable epimorphism $\underline{M} \rightarrow \mathcal{X}$ is called a *presentation* of \mathcal{X} . Given such a presentation, the (weak) fiber product $\underline{M} \times_{\mathcal{X}} \underline{M}$ is representable. I.e. there exists an object $\mathcal{G} \in \mathbf{DMan}$ such that $\underline{\mathcal{G}} \cong \underline{M} \times_{\mathcal{X}} \underline{M}$. In such a case we may say that the stack \mathcal{X} is *Dirac differentiable* or *presentable*.

The Dirac manifold \mathcal{G} inherits a collection of morphisms $\mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{m}, \mathbf{i}$ making \mathcal{G} into a groupoid object internal to \mathbf{DMan} . Furthermore, the \mathbf{s} and \mathbf{t} maps are induced by two projection morphisms

$$\mathrm{pr}_1, \mathrm{pr}_2 : \underline{M} \times_{\mathcal{X}} \underline{M} \rightarrow \underline{M}.$$

Consequently, we can conclude that as maps of stacks \mathbf{s} and \mathbf{t} must be epimorphisms and hence submersions. Therefore, \mathcal{G} has the structure of a D-Lie groupoid. The next proposition says that this gives an inverse correspondence to the functor $\mathcal{G} \mapsto \mathbf{B}\mathcal{G}$.

Proposition 5.4. *Suppose $p : M \rightarrow \mathcal{X}$ is a presentation of \mathcal{X} and \mathcal{G} is a D-Lie groupoid such that $\underline{\mathcal{G}} \cong M \times_{\mathcal{X}} M$ as CFGs. Then $\mathbf{B}\mathcal{G} \cong \mathcal{X}$.*

Proof. Our constructions thus far have given us all of the tools needed to prove this. Most standard proofs in the smooth setting go through here. For example, see the proof of Theorem 2.22 in [1]. \square

We also have the following theorem:

Theorem 5.5. *Suppose \mathcal{G} and \mathcal{H} are D-Lie groupoids. Then the following are equivalent:*

- (i) $\mathbf{B}\mathcal{G}$ and $\mathbf{B}\mathcal{H}$ are isomorphic.
- (ii) There exists a D-Lie groupoid $\tilde{\mathcal{H}}$ and weak equivalences $F : \tilde{\mathcal{H}} \rightarrow \mathcal{G}$ and $F : \tilde{\mathcal{H}} \rightarrow \mathcal{H}$.
- (iii) There exists a principal $(\mathcal{G}, \mathcal{H})$ -bibundle.

Proof. We use the same proof as in the proof of Theorem 2.26 in [1]. \square

Example 5.6 (The Stack of a Poisson Manifold). We saw earlier that there is a 1-1 correspondence between symplectic groupoids (\mathcal{G}, Ω) integrating a Poisson manifold (M, π) and target aligned D-Lie groupoid \mathcal{G} with a symplectic characteristic form. Furthermore, notice that Theorem 1.1 is an immediate corollary of Proposition 4.8 and Theorem 5.5.

Example 5.7 (A non-presentable Stack). One weakness of the category \mathbf{DMan} is that it lacks a terminal object. This is remedied by passing to stacks over \mathbf{DMan} . In fact, the category \mathbf{DMan} equipped with the identity projection is a terminal object in the 2-category of stacks. This is, perhaps, the simplest example of a stack over \mathbf{DMan} which does not admit a presentation.

We hope that this satisfies the reader that our notion of stack suitably captures the existing theory of symplectic Morita equivalences. In the next section, we will use what we have learned to study Morita equivalences infinitesimally.

6. INFINITESIMAL WEAK EQUIVALENCES

To treat infinitesimal weak equivalences of Poisson manifolds, we first make a few comments about general algebroids. We will say that a morphism of Algebroids $F : A \rightarrow B$ is *transverse* if it covers a smooth map $f : M \rightarrow N$ which is transverse to the foliation distribution of B . Our goal is to provide an infinitesimal criteria for a morphism of Lie algebroids to integrate to a weak equivalence of Lie groupoids. We begin with a simple lemma:

Lemma 6.1. *Suppose $F : A \rightarrow B$ is an algebroid morphism covering $f : M \rightarrow N$ and A and B are integrable. Then F integrates to a weak equivalence $\mathcal{F} : \Sigma(A) \rightarrow \Sigma(B)$ if and only if the following hold for all $x \in M$:*

- F is transverse;
- F induces a homeomorphism of orbit spaces $M/A \rightarrow N/B$;
- $\mathcal{F}_x : \Sigma(A)_x \rightarrow \Sigma(B)_x$ is an isomorphism.

Proof. The first two properties imply that \mathcal{F} is essentially surjective. We only need to show that \mathcal{F} is fully faithful. Suppose $f(x_1) = y_1$ and $f(x_2) = y_2$. Let $h \in \Sigma(B)$ be an arrow $h : y_1 \rightarrow y_2$. Since A and B have the same leaf space we know there exists $g \in \Sigma(A)$ such that $g : x_1 \rightarrow x_2$. Let $g' \in \Sigma(M)_{x_1}$ be such that $\mathcal{F}(g') = hF(g)^{-1}$. Then $h = F(g'g)$. This shows that \mathcal{F} is full. The morphism \mathcal{F} is faithful since $\mathcal{F}(g_1) = \mathcal{F}(g_2)$ if and only if $F(g_1g_2^{-1})$ is a unit, and therefore $g_1 = g_2$. \square

This tells us that in order to obtain an infinitesimal criteria for \mathcal{F} to be a weak equivalence, we need to understand under what conditions F_x is an isomorphism.

6.1. Monodromy. For a Lie algebroid A , the notation $\mathcal{G}(\mathfrak{g}_x)$ will denote the source simply connected integration of the isotropy Lie algebra of A at $x \in M$.

Definition 6.2. Let A be an integrable Lie algebroid. The *monodromy* of A at $x \in M$, $\mathcal{N}_x(A)$ is the kernel of the canonical map $\mathcal{G}(\mathfrak{g}_x) \rightarrow \Sigma(A)_x$.

If we think of elements of $\mathcal{G}(\mathfrak{g}_x)$ as \mathfrak{g}_x -paths modulo \mathfrak{g}_x -homotopy, the map $\mathcal{G}(\mathfrak{g}_x) \rightarrow \Sigma(A)_x$ given by passing to A -homotopy. $\mathcal{N}_x(A)$ fits into the following s.e.s.:

$$1 \longrightarrow \mathcal{N}_x(A) \longrightarrow \mathcal{G}(\mathfrak{g}_x) \longrightarrow \Sigma(A)_x^\circ \longrightarrow 1.$$

Given a morphism of algebroids $F : A \rightarrow B$ then there is an induced map $\tilde{F} : \mathcal{G}(\mathfrak{g}_x) \rightarrow \mathcal{G}(\mathfrak{g}_{f(x)})$ and for any $g \in \mathcal{G}(\mathfrak{g}_x) \in \mathcal{N}_x(A)$ we always have that $\tilde{F}(g) \in$

$\mathcal{N}_{f(x)}(B)$. Therefore, if $F : A \rightarrow B$ is a morphism of Lie algebroids, then for all $x \in M$ we get maps,

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{N}_x(A) & \longrightarrow & \mathcal{G}(\mathfrak{g}_x) & \longrightarrow & \Sigma(A)_x^\circ \longrightarrow 1 \\ & & \downarrow & & \downarrow \tilde{F} & & \downarrow \mathcal{F}_x \\ 1 & \longrightarrow & \mathcal{N}_{f(x)}(B) & \longrightarrow & \mathcal{G}(\mathfrak{g}_{f(x)}) & \longrightarrow & \Sigma(B)_{f(x)}^\circ \longrightarrow 1 \end{array},$$

which implies the next proposition.

Lemma 6.3. *Suppose $F : A \rightarrow B$ is a transverse morphism of Lie algebroids and $F' : A \rightarrow f^!B$ is an isomorphism. Let $\mathcal{F} : \Sigma(A) \rightarrow \Sigma(B)$ be the integration of F . Then $\mathcal{F}_x : \Sigma(A)_x^\circ \rightarrow \Sigma(B)_{f(x)}^\circ$ is an isomorphism if and only if $\tilde{F} : \mathcal{N}_x(A) \rightarrow \mathcal{N}_{f(x)}(B)$ is an isomorphism.*

To obtain a condition for an isomorphism of the full isotropy group, we observe that the group of connected components of $\Sigma(A)_x$ can be identified with $\pi_1(\mathcal{O}_x)$. In particular, for any algebroid A we have another short exact sequence:

$$1 \longrightarrow \Sigma(A)_x^\circ \longrightarrow \Sigma(A)_x \longrightarrow \pi_1(\mathcal{O}_x) \longrightarrow 1.$$

Again, if $F : A \rightarrow B$ is a morphism of Lie algebroids, we get maps,

$$\begin{array}{ccccccc} 1 & \longrightarrow & \Sigma(A)_x^\circ & \longrightarrow & \Sigma(A)_x & \longrightarrow & \pi_1(\mathcal{O}_x) \longrightarrow 1 \\ & & \downarrow \mathcal{F} & & \downarrow \mathcal{F} & & \downarrow f_* \\ 1 & \longrightarrow & \Sigma(B)_{f(x)}^\circ & \longrightarrow & \Sigma(B)_{f(x)} & \longrightarrow & \pi_1(\mathcal{O}_{f(x)}) \longrightarrow 1. \end{array}$$

We can immediately conclude the next result.

Lemma 6.4. *Suppose $F : A \rightarrow B$ is a transverse morphism of algebroids. Suppose further that $\mathcal{F}_x : \Sigma(A)_x \rightarrow \Sigma(B)_{f(x)}$ is an isomorphism of the connected component of the identity. Then \mathcal{F}_x is an isomorphism if and only if $(f|_{\mathcal{O}_x})_* : \pi_1(\mathcal{O}_x) \rightarrow \pi_1(\mathcal{O}_{f(x)})$ is an isomorphism.*

We can combine these two propositions to conclude that Theorem 1.2 holds. We call an algebroid morphism $F : A \rightarrow B$ satisfying (a-e) above a *weak equivalence*. Interestingly, The monodromy of A at x is also defined for non-integrable algebroids. In fact, the failure of $\mathcal{N}_x(A)$ to be discrete measures the failure of $\Sigma(A)$ to be smooth [7]. Therefore, this definition makes sense even when A and B are not integrable.

6.2. Weak equivalences of D-Lie algebroids. Theorem 1.2 has the following immediate consequence:

Corollary 6.5. *Suppose $F : A \rightarrow B$ is a morphism of integrable D-Lie algebroids. Then F integrates to a weak equivalence if and only if F satisfies the (a)-(e) of Theorem 1.2*

We define a morphism of D-Lie algebroids $F : A \rightarrow B$ to be a *weak equivalence* when F satisfies (a)-(e) above.

Example 6.6 (Poisson Manifolds). Suppose π_M and π_N are Poisson structures on M and N . Then the corresponding Dirac structures L_M and L_N are also D-Lie Algebroids. Therefore, we say a weak equivalence of Poisson manifolds $f : M \rightarrow N$ is a weak equivalence of their corresponding D-Lie Algebroids.

We can now prove Theorem 1.3.

Proof of Theorem 1.3. We first observe that given any weak equivalence $F : L_X \rightarrow L_M$, we have that:

$$L_X \cong f^! L_M := L_M \times_{\rho, df} X.$$

This construction is sometimes called the *pullback algebroid*. When L_M is integrable, then the pullback $f^! L_M$ is integrated by

$$f^! \Sigma(M) := X \times_{f, t} \Sigma(M) \times_{s, f} X.$$

Therefore, if $f : X \rightarrow M$ is a weak equivalence and M is integrable, then X is integrable. Hence, if $M \leftarrow X \rightarrow N$ are a pair of weak equivalences, then M and N are certainly Morita equivalent.

On the other hand, suppose M and N are Morita equivalent. Then there exists a symplectic bibundle $M \leftarrow P \rightarrow N$. Let $X := P$. Since the fibers of $\mathbf{t}^P : P \rightarrow M$ and $\mathbf{s}^P : P \rightarrow N$ are simply connected and \mathbf{s}^P and \mathbf{t}^P submersions:

$$(\mathbf{s}^P)^! \Sigma(M) \cong (\mathbf{t}^P)^! \Sigma(N) \cong \Sigma(X).$$

Therefore, \mathbf{s}^P and \mathbf{t}^P are the unit maps of weak equivalences of D-Lie groupoids. Therefore, they must be weak equivalences of Dirac manifolds. \square

APPENDIX A. PROOF OF LEMMA 3.2

Proof. Suppose \mathcal{G} with $\mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{m}, \mathbf{i}$ is a Lie groupoid and $(\mathbf{s}, \sigma), (\mathbf{t}, \tau), (\mathbf{u}, v), (\mathbf{i}, \iota)$ are morphisms in \mathbf{DMan} . This data constitutes a D-Lie groupoid if and only if the gauge equation associated to each groupoid axiom holds. In the table below, we have enumerated the axioms of a groupoid and computed the corresponding equations of 2-forms. We leave it to the reader to verify that these have been calculated correctly.

	Axiom	Domain	Gauge Part
(G1)	$\mathbf{s} \circ \mathbf{u} = \text{Id}_M$	M	$\mathbf{u}^* \sigma + v = 0$
(G2)	$\mathbf{s} \circ \mathbf{m} = \mathbf{s} \circ \text{pr}_2$	$\mathcal{G}^{(2)}$	$\mathbf{m}^* \sigma + \mu = \text{pr}_1^* \sigma + \text{pr}_2^* \sigma$
(G3)	$\mathbf{s} \circ \mathbf{i} = \mathbf{t}$	\mathcal{G}	$\mathbf{i}^* \sigma + \iota = \tau$
(G4)	$\mathbf{i} \circ \mathbf{u} = \mathbf{u}$	M	$\mathbf{u}^* \iota + v = v$
(G5)	$\mathbf{m} \circ ((\mathbf{u} \circ \mathbf{t}) \times \text{Id}_{\mathcal{G}}) = \text{Id}_{\mathcal{G}}$	\mathcal{G}	$((\mathbf{u} \circ \mathbf{t}) \times \text{Id})^* \mu = (\mathbf{u} \circ \mathbf{t})^* \sigma$
(G6)	$\mathbf{m} \circ (\text{Id}_{\mathcal{G}} \times (\mathbf{u} \circ \mathbf{s})) = \text{Id}_{\mathcal{G}}$	\mathcal{G}	$(\text{Id} \times (\mathbf{u} \circ \mathbf{s}))^* \mu = (\mathbf{u} \circ \mathbf{s})^* \tau$
(G7)	$\mathbf{m} \circ (\mathbf{i} \times \text{Id}_{\mathcal{G}}) = \mathbf{u} \circ \mathbf{s}$	\mathcal{G}	$(\mathbf{i} \times \text{Id})^* \mu - \mathbf{i}^* \sigma = \mathbf{s}^* v + \sigma$
(G8)	$\mathbf{m} \circ (\text{Id}_{\mathcal{G}} \times \mathbf{i}) = \mathbf{u} \circ \mathbf{t}$	\mathcal{G}	$(\text{Id} \times \mathbf{i})^* \mu - \mathbf{i}^* \tau = \mathbf{t}^* v + \tau$
(G9)	$\mathbf{m} \circ (\mathbf{m}(\text{pr}_1 \times \text{pr}_2) \times \text{pr}_3) = \mathbf{m} \circ (\text{pr}_1 \times \mathbf{m}(\text{pr}_2 \times \text{pr}_3))$	$\mathcal{G}^{(3)}$	see (A.2) below.

Now suppose we are supplied with 2-forms σ and τ satisfying (i) and (ii) from 3.2. Take the gauge equations from (G1-3) to be the definitions of v , μ and ι . Let $L_{\mathcal{G}} := \mathbf{s}^* L_M - \sigma$. We must show that \mathcal{G} and M together with $(\mathbf{s}, \sigma), (\mathbf{t}, \tau), (\mathbf{u}, v), (\mathbf{i}, \iota)$ constitutes a well defined D-Lie groupoid. Assumption (i) implies that (\mathbf{s}, σ) and (\mathbf{t}, τ) are well defined morphisms in \mathbf{DMan} . A careful calculation shows that the remaining maps are also morphisms of Dirac structures. It remains to show that the each gauge equation in the above table holds.

The equations from (G1-3) follow immediately by definition. The equation for (G4) holds since

$$\mathbf{u}^*(\iota) = \mathbf{u}^*(\tau - \sigma) = 0.$$

The last part follows since the unit map is isotropic with respect to any multiplicative form.

Next we show (G5) by first observing:

$$\begin{aligned} ((\mathbf{u} \circ \mathbf{t}) \times \text{Id})^* \mu &= ((\mathbf{u} \circ \mathbf{t}) \times \text{Id})^* (\text{pr}_1^* \sigma + \text{pr}_2^* \sigma - \mathbf{m}^* \sigma) \\ &= (\mathbf{u} \circ \mathbf{t})^* \sigma + \sigma - (\mathbf{m}((\mathbf{u} \circ \mathbf{t}) \times \text{Id}))^* \sigma \\ &= (\mathbf{u} \circ \mathbf{t})^* \sigma + \sigma - \sigma = (\mathbf{u} \circ \mathbf{t})^* \sigma \end{aligned}$$

It follows from the multiplicativity of $\tau - \omega$ that:

$$\mathbf{m}^* \tau + \mu = \text{pr}_1^* \tau + \text{pr}_2^* \tau \quad (\text{A.1})$$

By using this expression for μ we can show (G6) by a calculation essentially identical to (G5).

Next up, we show (G7):

$$\begin{aligned} (\mathbf{i} \times \text{Id})^* \mu - \mathbf{i}^* \sigma &= (\mathbf{i} \times \text{Id})^* (\text{pr}_1^* \sigma + \text{pr}_2^* \sigma - \mathbf{m}^* \sigma) - \mathbf{i}^* \sigma \\ &= \mathbf{i}^* \sigma + \sigma - (\mathbf{u} \circ \mathbf{s})^* \sigma - \mathbf{i}^* \sigma \\ &= -\mathbf{s}^* \mathbf{u}^* \sigma + \sigma = \mathbf{s}^* \iota + \sigma \end{aligned}$$

Since (G8) is similar we can proceed to (G9). The gauge equation for (G9) is

$$\begin{aligned} (\text{pr}_1 \times \text{pr}_2)^* \mu + (\mathbf{m} \circ (\text{pr}_1 \times \text{pr}_2) \times \text{pr}_3)^* \mu &= \\ (\text{pr}_2 \times \text{pr}_3)^* \mu + (\text{pr}_1 \times \mathbf{m} \circ (\text{pr}_2 \times \text{pr}_3))^* \mu. \end{aligned} \quad (\text{A.2})$$

If we apply the substitution $\mu = \text{pr}_1^* \sigma + \text{pr}_2^* \sigma - \mathbf{m}^* \sigma$ throughout, we get:

$$\begin{aligned} \text{pr}_1^* \sigma + \text{pr}_2^* \sigma - (\text{pr}_1 \times \text{pr}_2)^* \mathbf{m}^* \sigma + (\text{pr}_1 \times \text{pr}_2)^* \mathbf{m}^* \sigma + \text{pr}_3^* \sigma - \mathbf{A}_L^* \sigma &= \\ \text{pr}_2^* \sigma + \text{pr}_3^* \sigma - (\text{pr}_2 \times \text{pr}_3)^* \mathbf{m}^* \sigma + (\text{pr}_2 \times \text{pr}_3)^* \mathbf{m}^* \sigma + \text{pr}_1^* \sigma - \mathbf{A}_R^* \sigma. \end{aligned}$$

Here $\mathbf{A}_L, \mathbf{A}_R : \mathcal{G}^{(3)} \rightarrow \mathcal{G}$ are the left and right hand associativity maps. Since \mathcal{G} is a Lie groupoid and assumed to be associative, it follows immediately that (9) holds. \square

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